PROBLEM SET 2

Problem 1: Prove the following lemma of Fekete: if a_n is a sequence such that $a_{n+m} \leq a_n + a_m$, then $\lim a_n/n$ exists and is equal to $\inf a_n/n$.

Problem 2: Let (X, T) be a system such that $|\mathrm{Fix}(T^n)| < \infty$ for all n. Show that

$$\zeta_T(t) = \prod_{\text{finite } T-\text{orbits } \alpha} (1 - t^{|\alpha|})^{-1}$$

where $|\alpha|$ denotes the size of an orbit α.

Problem 3: Recall a system (X, T) is expansive if there exists $\epsilon > 0$ such that for all $x \neq y$, there exists $n \in \mathbb{Z}$ such that $d(T^n(x), T^n(y)) < \epsilon$. Show that if (X, T) is expansive, then $|\mathrm{Fix}(T^n)| < \infty$ for all $n \in \mathbb{N}$.

Problem 4: Suppose $\phi\colon (X, \sigma_X) \to (Y, \sigma_Y)$ is a factor map between subshifts for which there exists $M \geq 1$ such that $|\phi^{-1}(y)| \leq M$ for all $y \in Y$. Prove that $h(\sigma_X) = h(\sigma_Y)$.

Problem 5: Show that a system (X, T) is topologically transitive if and only if there exists $x \in X$ such that $\{T^n(x) \mid n \geq 0\}$ is dense in X.

Problem 6: Call a graph Γ *essential* if every vertex has at least one incoming edge and at least one outgoing edge.

(1) Show that if Γ is a finite directed graph then Γ has a unique subgraph Γ' which is essential, and $X_\Gamma = X_{\Gamma'}$.

(2) Show that an essential graph Γ is irreducible if and only if X_Γ is irreducible if and only if the adjacency matrix A_γ is irreducible.

(3) Show that if Γ is essential, then X_Γ is topologically mixing if and only A_Γ is primitive.

Problem 7: Find a subshift (X, σ_X) such that $h(\sigma_X) = \log 2$ but (X, σ_X) is not topologically conjugate to the full 2-shift.

Problem 8: Show that if (X, σ_X) is an irreducible shift of finite type, then periodic points are dense in X. Is this true for general shifts of finite type?

Bonus problem: Construct a subshift (X, σ_X) which has positive entropy but no periodic points.